Exercise 22

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + xy\,\mathbf{k},$$
$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le \pi$$

Solution

With the given parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^\pi \langle x(t), y(t), x(t)y(t)\rangle \cdot \frac{d}{dt}\langle \cos t, \sin t, t\rangle\,dt \\
&= \int_0^\pi \langle \cos t, \sin t, \cos t \sin t\rangle \cdot \langle -\sin t, \cos t, 1\rangle\,dt \\
&= \int_0^\pi [(\cos t)(-\sin t) + (\sin t)(\cos t) + (\cos t \sin t)(1)]\,dt \\
&= \int_0^\pi \cos t \sin t\,dt \\
&= \int_0^\pi \frac{1}{2}\sin 2t\,dt \\
&= \left(-\frac{1}{4}\cos 2t\right)\Big|_0^\pi \\
&= -\frac{1}{4}(\cos 2\pi - \cos 0) \\
&= 0.
\end{aligned}$$